

# UNIFORM ERROR ESTIMATES FOR NAVIER-STOKES FLOW WITH AN EXACT MOVING BOUNDARY USING THE IMMERSED INTERFACE METHOD

J. Thomas Beale

*Department of Mathematics, Duke University, Box 90320, Durham,  
North Carolina 27708, U.S.A.*

beale@math.duke.edu

## Abstract

We prove that uniform accuracy of almost second order can be achieved with a finite difference method applied to Navier-Stokes flow at low Reynolds number with a moving boundary, or interface, creating jumps in the velocity gradient and pressure. Difference operators are corrected to  $O(h)$  near the interface using the immersed interface method, adding terms related to the jumps, on a regular grid with spacing  $h$  and periodic boundary conditions. The force at the interface is assumed known within an error tolerance; errors in the interface location are not taken into account. The error in velocity is shown to be uniformly  $O(h^2 |\log h|^2)$ , even at grid points near the interface, and, up to a constant, the pressure has error  $O(h^2 |\log h|^3)$ . The proof uses estimates for finite difference versions of Poisson and diffusion equations which exhibit a gain in regularity in maximum norm.

## 1 Introduction

Recently there has been enormous development in numerical methods for fluid flow with moving boundaries or fluid-structure interaction. Often finite difference methods are used on a Cartesian grid which does not conform to the moving boundary. A separate representation is used for the boundary, and the effect of the boundary on the fluid must be included. For biological models practical applications have most often used the immersed boundary method [22] in which the force on the fluid from the boundary is spread to nearby grid points. Other methods maintain a sharp interface and are seen to attain about  $O(h^2)$  accuracy in the velocity. Generally these methods are carefully designed to control the truncation error, taking into account the location of the boundary relative to the grid cells. Here we focus on the immersed interface method ([13, 14, 15, 16, 17, 20, 25, 26, 27]) in which difference operators for the velocity and pressure are corrected where the stencil crosses the interface using jumps in the quantities and their derivatives. Closely related methods use ghost points or cut cells. With low to moderate Reynolds number, it is often observed in computations

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that the velocity error is about  $O(h^2)$  even when the truncation error is  $O(h)$  near the interface, but this phenomenon has not been explained, and understanding of the solution error has come mainly from numerical evidence.

In this work we estimate the errors in velocity and pressure, uniformly with respect to grid points, including those near the moving boundary, in a simple prototype problem using the immersed interface method. We neglect possible errors in the boundary location and concentrate on the errors in fluid variables brought about by the numerical treatment of the force from the moving boundary. Thus for a problem with coupled motion of the fluid and moving boundary we are only partially accounting for the errors. We verify analytically the principle that  $O(h)$  truncation error at the moving boundary can lead to uniform accuracy close to  $O(h^2)$ . In doing so we elucidate the minimal features necessary to achieve this accuracy. This result depends on the effect of diffusion with implicit time stepping and thus is significant at low Reynolds number.

We will always measure errors in maximum, or  $L^\infty$ , norm. One reason is that methods in use are generally designed to control maximum truncation errors near the moving boundary. A second reason is that the errors in the solution are likely to be largest near the boundary, and the most meaningful measure of the error is a uniform estimate. We use estimates derived in [3, 4] for discrete Poisson and diffusion equations with a gain in regularity in maximum norm. Although we have chosen to study the immersed interface method, we hope that the analytic technique introduced here will be suggestive for the larger class of related methods.

We first state the physical problem. We consider fluid flow in a rectangular region  $[-L, L]^d$  in dimension  $d = 2$  or  $3$ , with velocity  $u$  and pressure  $p$ , both periodic. We suppose the moving boundary or interface  $\Gamma$  is a closed curve in  $\mathbb{R}^2$  or closed surface in  $\mathbb{R}^3$ . We assume the density is constant and the Reynolds number is low to moderate, and for simplicity we set both to 1. The fluid flow is determined by the Navier-Stokes equations for the velocity and pressure, with a force exerted by the interface on the fluid,

$$u_t + u \cdot \nabla u + \nabla p = \Delta u + f \delta_\Gamma, \quad \nabla \cdot u = 0 \quad (1.1)$$

where  $f$  is the force density,  $\delta_\Gamma$  is the delta function restricting to the surface  $\Gamma$ , and  $\Delta = \nabla^2$  is the Laplacian. Equivalently, the equation holds away from  $\Gamma$  with zero force, and the velocity and pressure have the jumps across  $\Gamma$

$$[u] = 0, \quad \left[ \frac{\partial u}{\partial n} \right] = -f_{tan}, \quad f_{tan} \equiv f - (f \cdot n)n \quad (1.2)$$

$$[p] = f \cdot n, \quad \left[ \frac{\partial p}{\partial n} \right] = \nabla_\Gamma \cdot f_{tan} \quad (1.3)$$

Here  $n$  is the outward unit normal at  $\Gamma$ , and  $[p] = p_+ - p_-$  is the difference between the outside and inside values at  $\Gamma$ . (See e.g. [13, 15, 16, 17, 23, 26].)

The operator  $\nabla_\Gamma \cdot$  is the surface divergence; in  $\mathbb{R}^2$  it is just the arclength derivative. (E.g, see [2], (9.41.1) for the definition of  $\nabla_\Gamma \cdot$  and [26] for a thorough derivation of  $[\partial p / \partial n]$ .) The fact that the pressure  $p$  is periodic depends on the facts that

$$[u \cdot \nabla u] \cdot n = 0 \quad \text{and} \quad \int_\Gamma \left[ \frac{\partial p}{\partial n} \right] dS = 0 \quad (1.4)$$

At each  $t$ , the pressure has an indefinite constant; adding  $p_0(t)$  to  $p(x, t)$  does not change (1.1) or (1.3).

Typically  $\Gamma$  moves with the fluid velocity and the force  $f$  is determined from the configuration of  $\Gamma$ , depending on its material properties, e.g. elastic forces, so that  $f$  and  $\Gamma$  depend on the fluid variables. In this work we assume the location of  $\Gamma$  is known exactly and  $f$  is known within a certain error tolerance; see Theorem 1.1 below. Thus, for the full problem, we estimate only the part of the error in velocity and pressure from their direct computation while neglecting the influence of errors in  $\Gamma$ . With this qualification, the maximum errors in velocity and pressure in the scheme studied here are shown to be  $O(h^2|\log h|^2)$  and  $O(h^2|\log h|^3)$ , resp.

We discretize  $u$ ,  $p$  and their derivatives on a regular grid at points  $x_j = (j_1, j_2)h$  or  $x_j = (j_1, j_2, j_3)h$  with  $h = L/N$ . We use the usual centered differences for the discrete gradient  $\nabla_h$ , divergence  $\nabla_h \cdot$  and Laplacian  $\Delta_h$ . All are  $O(h^2)$  accurate for smooth functions, and thus at grid points where the stencil does not cross the interface  $\Gamma$ . In the immersed interface method, the differences are corrected at the irregular points using jumps at  $\Gamma$  for the variables and their partial derivatives. For example, for a function  $v(x)$ ,  $x \in \mathbb{R}$ , if  $x_{j-1}$ ,  $x_j$  are inside  $\Gamma$  and  $x_{j+1}$  is outside,  $\Gamma$  intersects the grid line at  $x^*$  with  $x_j \leq x^* \leq x_{j+1}$ , and  $h_+ = x_{j+1} - x^*$ , then

$$v_x(x_j) = \frac{v_{j+1} - v_{j-1}}{2h} - \frac{1}{2h} \left( [v] + h_+[Dv] + \frac{h_+^2}{2}[D^2v] \right) + O(h^2) \quad (1.5)$$

$$v_{xx}(x_j) = \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} - \frac{1}{h^2} \left( [v] + h_+[Dv] + \frac{h_+^2}{2}[D^2v] \right) + O(h) \quad (1.6)$$

provided  $v$  is  $C^3$  on each side, with similar but different formulas at  $x_{j+1}$ ; e.g., see [16, 27, 26]. Jumps in the first and second partial derivatives of  $u$  and  $p$ , needed for corrections here, can be found from (1.2), (1.3), as explained in the works cited.

Next we present the scheme to be analyzed. We choose a time step  $\tau = O(h)$  and compute the velocity  $u^n$  at time  $t_n = n\tau$ . In updating  $u^n$  to  $u^{n+1}$ , we will assume the force  $f$  and needed corrections are known up to time  $t_{n+1}$ . (If the interface  $\Gamma$  is updated explicitly, the simplest possibility, then  $\Gamma^{n+1}$  is found from  $u^n$ , and it determines the jumps at time  $t_{n+1}$ .) Assuming  $u^0$  is given, we start with

$$u^1 - u^0 = -\tau(u \cdot \nabla u)^0 - \tau \nabla p^0 + \tau \Delta u^{1/2} + \tau C_1 + \tau C_7 \quad (1.7)$$

Then, with  $u$  known up to time  $t_n$ ,  $n \geq 1$ , the new velocity  $u^{n+1}$  is found from

$$u^{n+1} - u^n = -\tau(u \cdot \nabla u)^{n+1/2} - \tau \nabla p^{n+1/2} + \tau \Delta u^{n+1/2} + \tau C_1 + \tau C_7 \quad (1.8)$$

We will describe the discretization of each term. Here  $C_1$  corrects the approximation  $u_t^{n+1/2} \approx (u^{n+1} - u^n)/\tau$  at grid points which are crossed by the interface during the interval  $t_n \leq t \leq t_{n+1}$ . This correction is a term proportional to  $[u_t]$ . Since the velocity is continuous at the interface, the material derivative  $u_t + u \cdot \nabla u$  is also continuous, so that  $[u_t] = -u \cdot [\nabla u]$ . (E.g. see [13], (33)-(35) or [26], Cor. 3.2.) Corrections to other terms at locations crossed by the interface during a time interval will be included in  $C_7$ , discussed later.

The advection term is extrapolated in time for  $n \geq 1$ ,

$$(u \cdot \nabla u)^{n+1/2} = \frac{3}{2}u^n \cdot \nabla_h u^n - \frac{1}{2}u^{n-1} \cdot \nabla_h u^{n-1} + C_2 \quad (1.9)$$

where  $C_2$  is the correction to the centered difference  $\nabla_h$ , determined from the jumps  $[Du]$  and  $[D^2u]$ , at each time  $t_n$  and  $t_{n-1}$  as in (1.5). Similarly

$$\Delta u^{n+1/2} = \frac{1}{2} (\Delta_h u^{n+1} + \Delta_h u^n) + C_3 \quad (1.10)$$

with corrections  $C_3$  to  $\Delta_h$  again determined from  $[Du]$ ,  $[D^2u]$  at each time as in (1.6).

For the pressure term we first compute the divergence of (1.9),

$$\nabla \cdot (u \cdot \nabla u)^{n+1/2} = \nabla_h \cdot (u \cdot \nabla u)^{n+1/2} + C_4 \quad (1.11)$$

where  $C_4$  corrects  $\nabla_h \cdot$  for the jumps in  $[Du]$ ,  $[D^2u]$ . We then solve the discrete Poisson problem

$$\Delta_h p^{n+1/2} = -\nabla_h \cdot (u \cdot \nabla u)^{n+1/2} - C_4 + C_5 - m \quad (1.12)$$

with periodic boundary conditions. Here  $C_5$  corrects  $\Delta_h p^{n+1/2}$  using the jumps  $[p]$ ,  $[Dp]$ ,  $[D^2p]$  and  $[u \cdot \nabla u]$ . The last term  $m$  is the mean value, or average, of  $-C_4 + C_5$  on the grid. It is subtracted so that the right side above has mean value zero and thus is in the range of  $\Delta_h$ ; we will see that the error resulting from  $m$  is not significant. The periodic solution of (1.12) has an indefinite constant; we choose it to have mean value zero. Finally,

$$\nabla p^{n+1/2} = \nabla_h p^{n+1/2} + C_6 \quad (1.13)$$

where  $C_6$  corrects  $\nabla_h$  using  $[p]$ ,  $[Dp]$ ,  $[D^2p]$ .

For each of the terms  $(u \cdot \nabla u)^{n+1/2}$ ,  $\nabla p^{n+1/2}$ ,  $\Delta u^{n+1/2}$  we compute separately at two time levels, including corrections. At a grid point crossed by the interface during the time interval we add a correction proportional to the jump in each quantity; see (14),(15) in [13]. These terms form  $C_7$ .

We will show that this scheme produces values of  $u$  and  $p$  that have accuracy slightly less than  $O(h^2)$  with certain assumptions. We summarize the conclusion as follows.

**Theorem 1.1.** *Suppose the exact solution of (1.1)-(1.3) is smooth in space-time on each side of the interface  $\Gamma$  for  $0 \leq t \leq T$ , and also  $\Gamma$  is smooth. We neglect any errors in  $\Gamma$ , and we assume that  $f$  and  $\nabla_{tan} f$  are known within maximum error  $O(h^2)$ . Then, with  $\tau/h$  constant and  $h$  sufficiently small,*

$$\max_{j,n} |u^{computed}(x_j, t_n) - u^{exact}(x_j, t_n)| \leq K_T h^2 |\log h|^2 \quad (1.14)$$

for  $t_n = n\tau \leq T$  and some constant  $K_T$  independent of  $h$ . The pressure  $p^n$  can be found as above at time  $t_n$  so that, for some constant  $p_0^n$ , depending on  $h$ ,  $p^n + p_0^n$  differs from the exact pressure with maximum error bounded by  $K_T h^2 |\log h|^3$ .

We have assumed periodic boundary conditions for the computational domain to avoid the serious issue of handling the boundary conditions, in order to focus on the accuracy near the interface. The difficult question of combining solid boundary conditions with the pressure

solution has been dealt with extensively for finite difference methods including projection methods (e.g. [6, 7, 8, 18]). We expect that in principle the two issues are separate, provided the interface is away from the outer boundary. The periodic condition is helpful for the analysis in that various operators commute in this case.

Often a MAC, or staggered, grid is used for velocity and pressure, in the present problem ([13, 25]) and others. It allows more natural treatment of computational boundaries. A second advantage is that the discrete version of the projection on divergence-free vector fields is an exact projection. On the other hand, the simplicity of a single grid, as in the present case, is a desirable advantage. We expect that a result similar to the present one would hold using a periodic MAC grid. A pressure increment scheme, updating the pressure rather than finding it from (1.12), might be used, as in [13, 17]. Such a scheme would be different from this one because of the effect of the discrete projection [1, 9] and the present analysis would not apply directly.

For temporal discretization of the diffusion we have chosen to use the Crank-Nicolson (CN) method since it is the most familiar second-order accurate method allowing time step  $O(h)$  with viscosity and since it is often used with interface simulations and with the projection method. CN has an important smoothing property in maximum norm, proved in [13], related to  $A$ -stability. This smoothing is largely responsible for the result stated above. However, other methods (called  $L$ -stable) such as BDF2 have better smoothing properties. The result proved here should also be true for these methods and perhaps can be improved.

A disadvantage with the use of a single grid is that  $(\nabla_h \cdot) \nabla_h \neq \Delta_h$ . Instead  $(\nabla_h \cdot) \nabla_h = \Delta_0$ , where  $\Delta_0$  is the “wide Laplacian”, a sum of second differences such as

$$v_{xx}(x_j) \approx (4h)^{-2} (v_{j-2} - 2v_j + v_{j+2}) \quad (1.15)$$

Consequently the discrete version  $\tilde{P} = I - \nabla_h (\Delta_h)^{-1} \nabla_h \cdot$  of the projection onto divergence-free vector fields is only an approximate projection, i.e.,  $\tilde{P}^2 \neq \tilde{P}$  (cf. [1, 9]). We will see that  $\tilde{P}$  enters in expressing the error in  $u$ . We will find that the exact discrete projection  $P_0$ , defined with  $(\Delta_0)^{-1}$  rather than  $(\Delta_h)^{-1}$ , is useful in our estimates.

In Sec. 2 we collect facts about difference operators in maximum norm relevant to this work. We discuss  $\tilde{P}$  and  $P_0$ . We give an estimate for a discrete Poisson problem with gain of regularity. We state estimates for CN time steps, also with gain in regularity. We state a lemma which allows a grid function near the interface to be estimated in a lower norm with a gain of a factor of  $h$ . In Sec. 3 we classify the truncation errors made by the exact solution in satisfying the scheme. We can allow errors which, for example, are first differences of quantities  $O(h^2)$ . In Sec. 4 we estimate the growth of errors in the velocity using the results of Secs. 2 and 3 and verify the conclusion above. Finally in the Appendix we give a criterion for boundedness of a discrete linear operator in maximum norm and show that a certain operator relating  $\tilde{P}$  and  $P_0$  is bounded.

Previous analysis of Cartesian grid methods with interfaces has concerned elliptic problems ([3, 16, 19]) and linear diffusion ([4], Sec. 8). The effect of discrete projections on accuracy with irregular boundaries was studied in [10]. The case of boundary conditions at irregular boundaries in Cartesian grids, rather than interfaces, is quite different and has a long history; e.g. see [21], Ch. 6. Analysis of finite difference methods for the Navier-Stokes equations usually assumes smooth, rather than piecewise smooth, solutions and uses  $L^2$  estimates.

We use the letter  $K$  for generic constants independent of  $h$ .  $D$  will be any first spatial derivative, and  $D_h$  any difference operator, not necessarily centered. However,  $\nabla_h$  in a gradient or divergence will always mean the centered difference.

## 2 Operators on periodic grids

We collect facts about several difference operators on periodic grid functions that will be used in the following arguments. Let  $\Omega_h$  be the set of grid points  $x_j = jh$  in  $\mathbb{R}^d$  where  $j$  is a  $d$ -tuple of integers with  $|j_\nu| \leq L$ ,  $1 \leq \nu \leq d$ , and let  $\mathcal{F}(\Omega_h)$  be the space of periodic grid functions on  $\Omega_h$ . We always use the maximum norm for such functions: For  $w \in \mathcal{F}(\Omega_h)$ ,  $\|w\| = \max_{x_j \in \Omega_h} |w(x_j)|$ . Correspondingly, for a linear operator  $\mathcal{L}$  on  $\mathcal{F}(\Omega_h)$  or a subspace,  $\|\mathcal{L}\| = \max \|\mathcal{L}w\|$  for  $\|w\| = 1$ . Of course we are primarily interested in how these norms depend on  $h$ . It will be helpful that the difference operators and inverses we deal with commute because of the periodicity.

The Fourier modes

$$e_k(x_j) = e^{ikjh\pi/L}, \quad k \in \mathbb{Z}^d, \quad -L/h < k_\nu \leq L/h \quad (2.1)$$

form a basis of  $\mathcal{F}(\Omega_h)$ . They are eigenfunctions for  $\Delta_h$  and the “wide Laplacian”  $\Delta_0$ ,

$$\Delta_h e_k = \sigma(kh) e_k, \quad \sigma(kh) = -\frac{4}{h^2} \sum_{\nu=1}^d \sin^2 \frac{kh\pi}{2L} \quad (2.2)$$

$$\Delta_0 e_k = \sigma_0(kh) e_k, \quad \sigma_0(kh) = -\frac{1}{h^2} \sum_{\nu=1}^d \sin^2 \frac{kh\pi}{L} \quad (2.3)$$

Evidently the null space of  $\Delta_h$  consists of constant functions, the multiples of  $e_0$ ; the null space  $\mathcal{N}_0$  of  $\Delta_0$  has dimension  $2^d$  and is spanned by  $e_k$  with each  $k_\nu = 0$  or  $L/h$ . (Cf. [1].) Each operator is invertible on the subspace of  $\mathcal{F}(\Omega_h)$  spanned by the remaining modes, which is also the range of the operator. We will call these subspaces  $X_h$  and  $X_0$ , respectively. Note that  $X_h = \{w \in \mathcal{F}(\Omega_h) : \sum_j w(x_j) = 0\}$ , the subspace with mean value zero.  $\Delta_h$  is invertible on  $X_h$ , and we will write  $(\Delta_h)^{-1}$  on  $X_h$ . Similarly we have  $(\Delta_0)^{-1}$  on  $X_0$ . If  $D_h$  is any centered first difference and  $w \in \mathcal{F}(\Omega_h)$  then  $D_h w \in X_0$  since  $D_h$  is zero on  $\mathcal{N}_0$ . Thus  $(\Delta_0)^{-1} D_h$  and  $(\Delta_h)^{-1} D_h$  are meaningful for any centered  $D_h$ .

Next we discuss the two discrete versions of the projection on divergence-free vector fields. The “exact discrete projection”, again using centered differences  $\nabla_h$ , is

$$P_0 v = v - \nabla_h (\Delta_0)^{-1} \nabla_h \cdot v \quad (2.4)$$

Since  $\nabla_h \cdot \nabla_h = \Delta_0$ ,  $\nabla_h \cdot P_0 = 0$ , and it follows that  $(I - P_0)P_0 = 0$ , or  $P_0^2 = P_0$  and  $(I - P_0)^2 = (I - P_0)$ ; that is,  $P_0$  and  $(I - P_0)$  are exact projections on  $\mathcal{F}(\Omega_h)$ .

The approximate projection  $\tilde{P}$  uses  $\Delta_h$  rather than  $\Delta_0$ ,

$$\tilde{P} v = v - \nabla_h (\Delta_h)^{-1} \nabla_h \cdot v \quad (2.5)$$

To relate the two, we write

$$\tilde{P} = I - \nabla_h \Delta_h^{-1} \nabla_h \cdot = P_0 + \nabla_h (\Delta_0^{-1} - \Delta_h^{-1}) \nabla_h \cdot \quad (2.6)$$

and

$$\tilde{P} - P_0 = \nabla_h(\Delta_h - \Delta_0)\Delta_h^{-1}\Delta_0^{-1}\nabla_h \cdot = (\Delta_h - \Delta_0)\Delta_h^{-1}(\nabla_h\Delta_0^{-1}\nabla_h \cdot) = A(I - P_0) \quad (2.7)$$

where  $A = (\Delta_h - \Delta_0)\Delta_h^{-1}$  so that

$$\tilde{P} = P_0 + A(I - P_0), \quad P_0\tilde{P} = P_0, \quad (I - P_0)\tilde{P} = A(I - P_0) \quad (2.8)$$

The following lemma, proved in the Appendix, tells us that  $A$  is bounded.

**Lemma 2.1.** *The operator  $A = (\Delta_h - \Delta_0)\Delta_h^{-1}$  on  $X_h$  has  $\|A\| \leq K$ , with  $K$  independent of  $h$ .*

We have estimates for the inverse Laplacians with gain of two discrete derivatives:

**Lemma 2.2.** *As operators on  $X_h$*

$$\|\Delta_h^{-1}\| \leq K_0, \quad \|D_h\Delta_h^{-1}\| \leq K_1, \quad \|D_h^2\Delta_h^{-1}\| \leq K_2 |\log h| \quad (2.9)$$

where  $D_h$  is any first difference operator and  $D_h^2$  is the product of any two, with constants independent of  $h$ . The same is true for  $\Delta_0^{-1}$  on  $X_0$  provided the operators  $D_h$  are centered differences.

The statement for  $\Delta_h$  is proved in [4], Cor. 3.2 and in an equivalent form in [3] The case of  $\Delta_0$  follows by applying the first case to subgrids with size  $2h$ . The  $|\log h|$  factor cannot be improved; this can be seen by example. We will use the fact that  $\nabla_h\Delta_0^{-1}$  is bounded. This elliptic estimate applies directly to the projections  $\tilde{P}$  and  $P_0$ , since we can write

$$(I - \tilde{P})v = \nabla_h(\Delta_h)^{-1}\nabla_h \cdot v = \nabla_h(\nabla_h \cdot \Delta_h^{-1})(v - \langle v \rangle) \quad (2.10)$$

where  $\langle v \rangle$  is the average of  $v$ , and similarly for  $P_0$ . Then from the lemma we have

$$\|\tilde{P}\| \leq K|\log h|, \quad \|P_0\| \leq K|\log h| \quad (2.11)$$

We will use estimates for the resolvent  $R$  of  $\Delta_h$  and the  $n$ th power  $S^n$  of the time-stepping operator  $S$  for Crank-Nicolson. We define

$$R = (I - \frac{\tau}{2}\Delta_h)^{-1}, \quad S = (I + \frac{\tau}{2}\Delta_h)(I - \frac{\tau}{2}\Delta_h)^{-1} \quad (2.12)$$

The following is proved in [4]; see (4.12), (4.13), (7.1), (7.2).

**Lemma 2.3.** *As operators on  $\mathcal{F}(\Omega_h)$ ,*

$$\|R\| \leq K_1, \quad \|D_h R\| \leq K_2 \tau^{-1/2}, \quad (2.13)$$

$$\|S^n\| \leq K_3, \quad \|D_h S^n R\| \leq K_4 (n\tau)^{-1/2} \quad (2.14)$$

where  $D_h$  is any first difference and the constants are independent of  $h$  and  $\tau$ .

We will need to know that a grid function on the *irregular* points near the interface is almost the discrete divergence of a function which is smaller by a factor of  $h$ . The following is proved as Lemma 8.1 in [4]. Related statements are Lemma 2.2 in [11], Lemma 2.10 in [24], and Lemmas 2.2 and 2.6 in [3].

**Lemma 2.4.** *Let  $\mathcal{T}_h = \{t_n = n\tau, 0 \leq t_n \leq T\}$  and assume  $\tau/h$  is constant. Let  $\mathcal{I}_h$  be a subset of  $\Omega_h \times \mathcal{T}_h$  such that each  $(x_j, t_n) \in \mathcal{I}_h$  is within  $O(h)$  of  $\Gamma(t_n)$ . Let  $\varphi$  be a periodic function on  $\Omega_h \times \mathcal{T}_h$  which is zero outside of  $\mathcal{I}_h$ . Then there are periodic grid functions  $\Phi_\nu$ ,  $0 \leq \nu \leq d$  so that*

$$\varphi = \Phi_0 + \sum_{\nu=1}^d D_\nu^- \Phi_\nu \quad \text{and} \quad \|\Phi_\nu\| \leq Kh\|\varphi\| \quad (2.15)$$

with some constant  $K$  independent of  $h$ , where  $D_\nu^-$  is the backward difference in direction  $\nu$ , and the norms are the maximum over  $t_n \leq T$  as well as  $jh \in \Omega_h$ .

Lemma 2.4 applies directly to the correction terms and the remaining truncation errors near the interface, since they occur only at grid points within  $O(h)$  of  $\Gamma$ . If, for example,  $\varphi$  is function of  $O(h)$  on the irregular points, we express the conclusion briefly as  $\varphi = O(h^2) + D_h O(h^2)$  or  $\varphi = (I + D_h)O(h^2)$ . We might combine this with Lemma 2.3 and use the fact that the operators commute to conclude that  $\|S^n R\varphi(\cdot, t_k)\|$  is  $O(h^2(n\tau)^{-1/2})$ .

### 3 Consistency Estimates

We estimate the error the exact solution makes in satisfying the scheme (1.7)–(1.13). We will denote the exact velocity and pressure as  $v$  and  $q$  to distinguish them from the computed quantities  $u$  and  $p$ . We will verify that the error has the form  $O(h^2|\log h|) + D_h O(h^2|\log h|)$ ; the second term means a difference operator applied to a grid function with the specified bound.

**Lemma 3.1.** *For the exact velocity  $v$  and pressure  $q$  we have for  $n \geq 1$*

$$v^{n+1} - v^n = -\tau(v \cdot \nabla v)^{n+1/2} - \tau \nabla q^{n+1/2} + \tau \Delta v^{n+1/2} + \tau C_1 + \tau C_7 + \tau \varepsilon^n \quad (3.1)$$

where the quantities on the right are computed as in (1.9)–(1.13) from  $v$  at times  $t_{n-1}, t_n, t_{n+1}$  with correction terms. For  $n = 0$ ,

$$v^1 - v^0 = -\tau(v \cdot \nabla v)^0 - \tau \nabla q^0 + \tau \Delta v^{1/2} + \tau C_1 + \tau C_7 + \tau \varepsilon^0 \quad (3.2)$$

Here  $\varepsilon^n = E_0^n + D_h E_1^n$  with  $E_k^n = O(h^2|\log h|)$  for  $n \geq 1$ ,  $k = 0, 1$ , and  $\varepsilon^0 = O(h|\log h|)$ .

If  $v$  and  $q$  were smooth across the interface, the scheme would have  $O(h^2)$  truncation error. To verify the statement, we will consider the corrections at the irregular points near the interface and the remaining errors. Since corrections are made to first order accuracy, a typical truncation error  $\eta$  on the whole grid will have the form

$$\eta = \begin{cases} O(h) & \text{at an irregular point} \\ O(h^2) & \text{at a regular point} \end{cases} \quad (3.3)$$



Because of Lemma 2.4, we then have  $\eta = O(h^2) + D_h O(h^2)$ .

To estimate the errors, we first assume the force  $f$  is known exactly, so that the corresponding jumps in  $v$ ,  $q$ , and their derivatives are exact. Later we consider the effect of errors in  $f$ . We will use the superscript  $ex$  to denote exact quantities at time  $t_{n+1/2}$ , to distinguish them from quantities computed in the scheme from the exact  $v$ . The error  $(v^{n+1} - v^n)/\tau - C_1 - (v_t)^{ex}$  has the form (3.3) since  $C_1$  corrects for  $[u_t]$  if the interface crosses the grid point, leaving a remainder at such a point of  $O(\tau) = O(h)$ . In dealing with other terms we will first neglect such crossings and return to them afterward.

The error in the advection term,

$$\frac{3}{2}v^n \cdot \nabla_h v^n - \frac{1}{2}v^{n-1} \cdot \nabla_h v^{n-1} + C_2 - (v \cdot \nabla v)^{ex} = O(h^2) \quad (3.4)$$

uniformly, since at an irregular point the correction  $C_2$  uses  $[Dv]$ ,  $[D^2v]$ , leaving remainder  $O(h^3/h)$ . Similarly, the error  $\frac{1}{2}(\Delta_h v^n + \Delta_h v^{n+1}) + C_3 - (\Delta v)^{ex}$  has the form (3.3) since the remaining error at an irregular grid point after correcting with  $C_3$  is  $O(h^3/h^2)$ .

Because of (3.4), the error in the divergence

$$\varepsilon_4 = \nabla_h \cdot (v \cdot \nabla_h v)^{n+1/2} + C_4 - \nabla \cdot (v \cdot \nabla v)^{ex} \quad (3.5)$$

has the form (3.3), with the correction  $C_4$  similar to  $C_3$ . For the exact pressure  $q^{ex}$  we have  $\Delta q^{ex} = -\nabla \cdot (v \cdot \nabla v)^{ex}$ , with the prescribed jumps in  $p$  and  $\partial p / \partial n$ , so that

$$\Delta_h q^{ex} = -\nabla \cdot (v \cdot \nabla v)^{ex} + C_5 + \varepsilon_5 \quad (3.6)$$

where  $C_5$  corrects the discrete Laplacian with  $[p]$ ,  $[Dp]$ ,  $[D^2p]$ ,  $[u \cdot \nabla u]$ , and  $\varepsilon_5$  has the form (3.3). Combining (3.5) and (3.6) we have

$$\Delta_h q^{ex} = -\nabla_h \cdot (v \cdot \nabla_h v)^{n+1/2} - C_4 + C_5 + \varepsilon_4 + \varepsilon_5 \quad (3.7)$$

The pressure  $q^h$  corresponding to the scheme is the solution with mean value zero of the similar equation

$$\Delta_h q^h = -\nabla_h \cdot (v \cdot \nabla_h v)^{n+1/2} - C_4 + C_5 - m \quad (3.8)$$

where  $m$  is the average of  $-C_4 + C_5$  over the periodic box. We check that  $m$  is  $O(h^2)$ : In (3.7) the averages of  $\Delta_h q^{ex}$  and the  $\nabla_h \cdot$  term are zero, since they are differences. Because  $\varepsilon_4 + \varepsilon_5$  has the form (3.3), and the number of irregular points is  $O(h^{1-d})$ , its average is  $O(h^2)$ , and therefore the same is true for  $-C_4 + C_5$ . Now the right sides of (3.7), (3.8) differ by an error of the form (3.3), and it follows from Lemma 2.2 that  $q^h - (q^{ex} - q_0^{ex}) = O(h^2)$  uniformly, where  $q_0^{ex}$  is the mean value of  $q^{ex}$  on the grid. Then  $\nabla_h q^h - \nabla_h q^{ex} = D_h O(h^2)$ . Also  $\nabla_h q^{ex} + C_6 - \nabla q^{ex} = O(h)$  at irregular points, and thus is of the form (3.3). With  $\nabla q^{n+1/2} = \nabla_h q^h + C_6$ , we combine the last two estimates to conclude that  $\nabla q^{n+1/2} - \nabla q^{ex}$  is also of the form  $(D_h + I)O(h^2)$ .

For each of the terms  $(v \cdot \nabla v)^{n+1/2}$ ,  $\nabla q^{n+1/2}$ ,  $\Delta v^{n+1/2}$ , at a grid point crossed by the interface during the time interval, there is an additional correction with the jump in the quantity. The remaining error in time discretization at such a point is  $O(\tau) = O(h)$ , and this error again has the form (3.3).

We now consider the effect of errors in the force  $f$ . Suppose at first we assume only that the error in  $f$  is  $O(h^2)$ . Then the errors in  $[Du]$ ,  $[D^2u]$ , determined from (1.2), are at most

$O(h^2)$  and  $O(h)$ . Then, for example, the error in  $C_3$  is  $O(h^{-2} \cdot h \cdot h^2) = O(h)$ , which is again of the form (3.3) since it occurs only at irregular points. Similarly the error in correcting  $\Delta_h u$  where the interface crosses is  $O(h)$ . The error in  $C_1$  or  $C_2$  is  $O(h^2)$ , and in  $C_4$  it is  $O(h)$ . These are no larger than corresponding errors already estimated.

It seems that for the pressure we need the further assumption that the error in  $\nabla_{tan} f$  is  $O(h^2)$ . The errors in  $[p]$ ,  $[Dp]$ ,  $[D^2p]$  are then  $O(h^2)$ ,  $O(h^2)$ , and  $O(h)$ , respectively. The errors in  $[Dp]$ ,  $[D^2p]$  contribute an error to  $C_5$  which is  $O(h)$ . The error in  $[p]$  contributes an error which could be  $O(1)$ . However, the form of  $\Delta_h p$  (1.6) is such that the correction from  $[p]$  enters  $C_5$  as a difference, and consequently this error in  $C_5$  has the form  $D_h b$  for  $b = O(h)$ . (The difference structure for this correction was noted in [14].) This term  $b$  at irregular points has the form  $(D_h + I)O(h^2)$ , according to Lemma 2.4, and thus the error in  $C_5$  and  $\Delta_h p$  is  $(D_h^2 + D_h)O(h^2)$ . By Lemma 2.2 the resulting error in  $p$ , and the contribution to  $q^h - q^{ex}$  above, is  $O(h^2 |\log h|)$ , slightly worse than before. This error term leads to the log factor in the statement of the lemma.

For  $n = 0$  the accuracy of the extrapolation in  $v \nabla \cdot v + \nabla q$  is only  $O(h)$ . Since  $D_h O(h^2) = O(h)$ ,  $\varepsilon^0$  is at most  $O(h |\log h|)$ .

## 4 The error in the solution

We estimate the growth of the error in the velocity. We set  $w^n = u^n - v^n$ . Rather than estimate the maximum of  $w$  directly, it seems better to estimate separately  $y^n \equiv P_0 w^n$  and  $z^n \equiv (I - P_0)w^n$ . Of course  $\|w^n\| \leq \|y^n\| + \|z^n\|$ , but we cannot bound  $\|y^n\|$  by  $\|w^n\|$  because of the  $\log h$  factor in the bound (2.11) for the projection. We will see that  $P_0$  is useful in estimating the nonlinear terms.

We begin by subtracting the equations (1.8) and (3.1) for  $u^{n+1}$  and  $v^{n+1}$ . The corrections  $C_1, C_2, C_3, C_6, C_7$  cancel, and the the advection terms give us

$$g \equiv (u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2} \quad (4.1)$$

where we now use the superscript  $n + 1/2$  as a shorthand for the extrapolation in (1.9). The pressures  $p$  and  $q^h$  are defined by the similar equations (1.12), (3.8), so that  $\Delta_h(p - q^h) = -\nabla_h \cdot g$  and the gradient term in the equation is

$$\nabla_h p^{n+1/2} - \nabla_h q^{n+1/2} = -\nabla_h \Delta_h^{-1} \nabla_h \cdot g^{n+1/2} = -(I - \tilde{P})g^{n+1/2} \quad (4.2)$$

We can then combine terms to get

$$(u \cdot \nabla_h u)^{n+1/2} - (v \cdot \nabla_h v)^{n+1/2} + \nabla_h p^{n+1/2} - \nabla_h q^{n+1/2} = \tilde{P}g^{n+1/2} \quad (4.3)$$

and thus the equation for  $w$  has the simple form

$$w^{n+1} - w^n = -\tau \tilde{P}g^{n+1/2} + (\tau/2)(\Delta_h w^{n+1} + \Delta_h w^n) + \tau \varepsilon^n \quad (4.4)$$

where  $\varepsilon^n$  is the error in the  $v$ -equation (3.1). We introduce the operators  $R$  and  $S$  from (2.12) and rewrite (4.4) as

$$w^{n+1} = S w^n - \tau R \tilde{P}g^{n+1/2} + \tau R \varepsilon^n \quad (4.5)$$

For  $n = 0$ ,  $w^0 = 0$  and  $g^0 = 0$  since  $u^0 = v^0$ , and

$$w^1 = \tau R \varepsilon^0 \quad (4.6)$$

We obtain separate equations for  $y = P_0 w$  and  $z = (I - P_0)w$  by applying  $P_0$  and  $(I - P_0)$  through the  $w$ -equation and using the identities (2.8) for  $P_0 \tilde{P}$  and  $(I - P_0) \tilde{P}$ . We get

$$y^{n+1} = S y^n - \tau R P_0 g^{n+1/2} + \tau P_0 R \varepsilon^n \quad (4.7)$$

and

$$z^{n+1} = S z^n - \tau R A (I - P_0) g^{n+1/2} + \tau (I - P_0) R \varepsilon^n \quad (4.8)$$

and for  $n = 0$

$$y^1 = \tau P_0 R \varepsilon^0, \quad z^1 = \tau (I - P_0) R \varepsilon^0 \quad (4.9)$$

To estimate the growth of the error, we define

$$\delta^n = \max_{1 \leq m \leq n} (\|y^m\| + \|z^m\|), \quad n \geq 1 \quad (4.10)$$

We will prove by induction that

$$\delta^n \leq K_T h^2 (\log h)^2, \quad n\tau \leq T \quad (4.11)$$

for some constant  $K_T$  and for  $h$  sufficiently small. We will then have verified the estimate (1.14) for the error  $w$  in velocity. Once the estimate is proved for some  $n$ , it follows that  $\|D_h w^m\| \leq 1$  for  $m \leq n$  and for sufficiently small  $h$ ; we will use this below for the nonlinear term in the error.

To estimate  $g^{n+1/2}$  we write  $g = g_1 + g_2 + g_3$  with

$$g_1 = v \cdot \nabla_h w, \quad g_2 = w \cdot \nabla_h v, \quad g_3 = w \cdot \nabla_h w \quad (4.12)$$

It will be important that  $D_h v$  is uniformly bounded for any first difference  $D_h$  since  $v$  is continuous at the interface. We will use the notation  $B(w)$  for any bounded linear operator applied to  $w$ ; that is,  $\|B(w)\| \leq K\|w\|$  with constant  $K$  independent of  $h$ . Thus, for example, we can write the difference of a product  $v_j w_i$  as

$$D_h(v_j w_i) = v_j D_h w_i + B(w) \quad (4.13)$$

since  $D_h v$  is bounded. Then for  $g_1$  and  $g_2$  we have

$$g_1^{n+1/2} = D_h B(w^n) + B(w^n) + D_h B(w^{n-1}) + B(w^{n-1}), \quad g_2^{n+1/2} = B(w^n) + B(w^{n-1}) \quad (4.14)$$

For the nonlinear term, we can assume by induction, as remarked above, that  $\|D_h w^n\| = O(1)$ , and the same for  $\|w^{n-1}\|$ , so that  $\|g_3\| \leq K(\|w^n\| + \|w^{n-1}\|)$ .

The main difficulty in estimating  $y^n$  is the effect of  $P_0$  on  $g$ , since  $P_0$  has norm  $O(|\log h|)$ . Applying it directly would lose stability. Since we have already estimated  $g$ , it is equivalent to estimate  $P_0 g$  or  $(I - P_0)g$ , and we choose the latter,  $(I - P_0)g = \nabla_h \Delta_0^{-1} \nabla_h \cdot g$ . We note, using Lemma 2.2, that  $\nabla_h \Delta_0^{-1}$  is a bounded operator, since  $\nabla_h$  is a centered difference, and for some terms in  $g$  it will be enough to estimate the divergence.

With  $g$  as in (4.12), we begin with  $g_1$ . Writing  $(I - P_0)g_1 = (I - P_0)(v \cdot \nabla_h y + v \cdot \nabla_h z)$ , we treat the first term by using  $\nabla_h \cdot y = 0$ . We apply  $\nabla_h \cdot$  to  $v \cdot \nabla_h y$  obtaining (with sum over  $j$ , and  $D_j$  the centered difference for  $x_j$ ,  $1 \leq j \leq d$ ),

$$\nabla_h \cdot (v_j D_j y) = D_j \nabla_h \cdot (v_j y) + \nabla_h \cdot B(y) = D_j (v_j \nabla_h \cdot y) + (D_j + \nabla_h \cdot) B(y) = 0 + D_h B(y) \quad (4.15)$$

so that by the remark above

$$(I - P_0)(v \cdot \nabla_h y) = D_h B(y) \quad (4.16)$$

For the second term in  $g_1$ ,  $z$  is in the range of  $(I - P_0)$  and thus is a discrete gradient, so that  $D_j z_i = D_i z_j$ . Then the divergence is (with sum over  $i, j$ )

$$D_i (v_j D_j z_i) = D_i (v_j D_i z_j) = D_i^2 (v_j z_j) + D_i B(z) = \Delta_0 (v_j z_j) + D_i B(z) \quad (4.17)$$

Then  $(I - P_0)(v \cdot \nabla_h z) = \nabla_h \Delta_0^{-1} \Delta_0 (v_j z_j) + D_h B(z) = \nabla_h (v_j z_j) + D_h B(z) = D_h B(z)$ .

The term  $(I - P_0)g_2$  has the form  $\nabla_h \Delta_0^{-1} \nabla_h \cdot (w \cdot \nabla_h v) = D_h B(w)$ . For  $g_3$ , again by induction  $\nabla_h w = O(1)$ , and we can treat  $(I - P_0)g_3$  like  $(I - P_0)g_2$ . In summary we have shown that

$$P_0 g^{n+1/2} = \Phi_0^n + D_h \Phi_1^n + \Phi_0^{n-1} + D_h \Phi_1^{n-1} \quad (4.18)$$

where

$$\|\Phi_k^m\| \leq K(\|y^m\| + \|z^m\|), \quad k = 0, 1; \quad m = n-1, n \quad (4.19)$$

and  $(I - P_0)g$  has the same form.

We are now ready to prove (4.11) by induction. Since  $\varepsilon^0 = O(h|\log h|)$ , it is evident from (4.9) that (4.11) holds for  $n = 1$ . We assume it is true for  $n$  and prove it for  $n + 1$ . Here and below we use the fact that  $\|P_0\| \leq K|\log h|$ . From the  $y$ -equation (4.7) we have

$$y^{n+1} = -\tau \sum_{\ell=1}^n S^{n-\ell} R P_0 g^{\ell+1/2} + \tau \sum_{\ell=0}^n S^{n-\ell} R P_0 \varepsilon^\ell \equiv \Sigma_1 + \Sigma_2 \quad (4.20)$$

For  $\Sigma_1$  we use (4.18), (4.19) and (2.14) to estimate for  $\ell \leq n-1$

$$|S^{n-\ell} D_h R \Phi_1^\ell| \leq K((n-\ell)\tau)^{-1/2} \delta^\ell \quad (4.21)$$

and similarly for other terms, while for  $\ell = n$  we use (2.13) to get

$$|D_h R \Phi_1^n| \leq K\tau^{-1/2} \delta^n \quad (4.22)$$

Then

$$|\Sigma_1| \leq K_1 \left( \sum_{\ell=1}^{n-1} ((n-\ell)\tau)^{-1/2} \delta^\ell \tau + \tau^{-1/2} \delta^n \tau \right) \quad (4.23)$$

For  $\Sigma_2$  we recall from Sec. 3 that  $\varepsilon^n = (I + D_h)O(h^2|\log h|)$  for  $n \geq 1$ , and we again use (2.14) for  $1 \leq \ell \leq n-1$  and (2.13) for  $\ell = n$ . We treat  $\ell = 0$  separately using  $\varepsilon^0 = O(h|\log h|)$  and (2.14). Then  $|\Sigma_2|$  is bounded by a constant times

$$\sum_{\ell=1}^{n-1} ((n-\ell)\tau)^{-1/2} h^2 (\log h)^2 \tau + \tau^{-1/2} h^2 (\log h)^2 \tau + h (\log h)^2 \tau \leq K h^2 (\log h)^2 \quad (4.24)$$

since the sum approximates an integrable function of time. Estimates for  $z^{n+1}$  are entirely similar, since  $A$  is bounded independent of  $h$ , according to Lemma 2.1, and adding the two inequalities gives

$$\delta^{n+1} \leq K_1 \sum_{\ell=1}^{n-1} ((n-\ell)\tau)^{-1/2} \delta^\ell \tau + K_1 \tau^{-1/2} \delta^n \tau + K_2 h^2 (\log h)^2 \quad (4.25)$$

To simplify this we use Hölder's inequality to obtain

$$(\delta^{n+1})^3 \leq K'_1 \left( \sum_{\ell=1}^{n-1} ((n-\ell)\tau)^{-2/3} \tau + \tau^{-2/3} \tau \right)^2 \sum_{\ell=1}^n (\delta^\ell)^3 \tau + K'_2 (h^2 (\log h)^2)^3 \quad (4.26)$$

The first sum is uniformly bounded, and the inequality has the form

$$\kappa^{n+1} \leq A \sum_{\ell=1}^n \kappa^\ell \tau + B \quad (4.27)$$

with  $\kappa^n = (\delta^n)^3$  and  $B = (h^2 (\log h)^2)^3$ . It follows easily from this and the estimate for  $\delta^1$  that  $\delta^m$  has the bound (4.11) for  $m \leq n+1$ , provided  $(n+1)\tau \leq T$ .

Finally we discuss the accuracy of the pressure. We can compute the pressure  $p^n$  at time  $t_n$  as in (1.12) using  $(u \cdot \nabla_h u)^n$  and the corrections  $C_4, C_5$ . As in Sec. 3,  $q^h - (q - q_0) = O(h^2 |\log h|)$ , where  $q$  is the exact pressure,  $q^h$  is computed from the exact velocity  $v$  and corrections as in (3.8) at time  $t_n$ , and  $q_0$  is the mean value of  $q$  on the grid. Combining the estimate (4.11) or (1.14) for  $w$  with (4.14) for  $g$ , we now have  $g = (D_h + I)O(h^2 |\log h|^2)$ . Lemma 2.2 then gives

$$p^n - q^h = \Delta_h^{-1} \nabla_h \cdot (D_h + I)O(h^2 |\log h|^2) = O(h^2 |\log h|^3) \quad (4.28)$$

Thus  $p^n - (q - q_0) = O(h^2 |\log h|^3)$ , as stated.

## A Appendix

We prove a simple criterion for the boundedness of Fourier multiplier operators on periodic grid functions in maximum norm. We use this statement to prove Lemma 2.1. Related statements are given in [5], Ch.1 and [12], Sec. 2.5.1.

We will assume  $L = \pi$  for convenience and  $\pi/h$  is an integer. Let  $I_d$  be the set of integer  $d$ -tuples  $j$  with  $|j_\nu| \leq \pi/h$ ,  $1 \leq \nu \leq d$ . Then  $\Omega_h = hI_d$  and  $\mathcal{F}(\Omega_h)$  is the space of periodic grid functions. For  $\varphi \in \mathcal{F}(\Omega_h)$  we have the discrete Fourier transform and inverse

$$\hat{\varphi}(k) = \sum_{j \in I^d} \varphi(jh) e^{-ikjh}, \quad \varphi(jh) = (2\pi)^{-d} \sum_{k \in I^d} \hat{\varphi}(k) e^{ikjh} h^d \quad (A.1)$$

and the isometry

$$\sum_{j \in I^d} |\varphi(jh)|^2 = (2\pi)^{-d} \sum_{k \in I^d} |\hat{\varphi}(k)|^2 h^d. \quad (A.2)$$

**Lemma A.1.** Suppose an operator  $A$  is defined on  $\mathcal{F}(\Omega_h)$  by

$$(A\varphi)^\wedge(k) = \sigma(kh)\hat{\varphi}(k) \quad (\text{A.3})$$

where  $\sigma$  is a function of  $\xi \in \mathbb{R}^d$ , with period  $2\pi$  in each  $\xi_\nu$ ,  $1 \leq \nu \leq d$ . Let  $\|A\|$  be the norm of  $A$  as an operator on  $\mathcal{F}(\Omega_h)$  with maximum norm. Then

$$\|A\| \leq K \left( \sum_{k \in I^d} (|(D_\nu^+)^s \sigma(kh)|^2 + |\sigma(kh)|^2) h^d \right)^{d/4s} \left( \sum_{k \in I^d} |\sigma(kh)|^2 h^d \right)^{(1-d/2s)/2} \quad (\text{A.4})$$

where  $s$  is an integer,  $s > d/2$ ,  $(D_\nu^+)^s$  is the forward divided difference operator in direction  $\nu$ , a sum over  $1 \leq \nu \leq d$  is implied, and  $K$  is independent of  $h$ .

*Proof.* We can write  $A$  as a discrete convolution with the inverse transform  $a_j$  of  $\sigma(kh)$ ,

$$A\varphi = \sum_{\ell \in I^d} a_{j-\ell} \varphi(\ell h), \quad a_j = (2\pi)^{-d} \sum_{k \in I^d} \sigma(kh) e^{ikjh} h^d \quad (\text{A.5})$$

so that the operator norm of  $A$  on  $\mathcal{F}(\Omega_h)$  has the bound

$$\|A\| \leq \sum_{j \in I^d} |a_j|. \quad (\text{A.6})$$

We will temporarily extend this sum to all  $j \in \mathbb{Z}^d$  with  $a_j = 0$  for  $j \notin I^d$  and estimate as in the proof of Sobolev's theorem and particularly as in [5], Ch. 1, Thm. 3.1. With  $R$  to be chosen we have

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} |a_j| &= \sum_{|j| \geq R} |a_j| |j|^s |j|^{-s} + \sum_{|j| < R} |a_j| \\ &\leq \left( \sum_{|j| \geq R} |j|^{-2s} \right)^{1/2} \left( \sum_{|j| \geq R} |a_j|^2 |j|^{2s} \right)^{1/2} + \left( \sum_{|j| \leq R} 1 \right)^{1/2} \left( \sum_{|j| \leq R} |a_j|^2 \right)^{1/2} \\ &\leq \left( \int_{|x| \geq R - \sqrt{d}} |x|^{-2s} dx \right)^{1/2} M_1 + \left( \int_{|x| \leq R + \sqrt{d}} dx \right)^{1/2} M_0 \\ &\leq K_1 (R - \sqrt{d})^{(-2s+d)/2} M_1 + K_0 (R + \sqrt{d})^{d/2} M_0 \quad (\text{A.7}) \end{aligned}$$

where

$$M_0^2 = \sum_{j \in \mathbb{Z}^d} |a_j|^2, \quad M_1^2 = \sum_{j \in \mathbb{Z}^d} |a_j|^2 (|j|^{2s} + 1). \quad (\text{A.8})$$

We choose  $R = \sqrt{d} + (M_1/M_0)^{1/s}$ , so that  $R \geq \sqrt{d} + 1$  and  $(R + \sqrt{d})/(R - \sqrt{d})$  is bounded above. Then both terms at the end of (A.7) have  $M_1^{d/2s} M_0^{1-d/2s}$ , and the inequality simplifies to

$$\sum_{j \in \mathbb{Z}^d} |a_j| \leq K \left( \sum_{j \in \mathbb{Z}^d} |a_j|^2 (1 + |j|^{2s}) \right)^{d/4s} \left( \sum_{j \in \mathbb{Z}^d} |a_j|^2 \right)^{(1-d/2s)/2}. \quad (\text{A.9})$$

Next we relate  $|j|^s a_j$  to  $(D_\nu^+)^s \sigma(kh)$ . A summation by parts using periodicity shows that

$$(2\pi)^{-d} \sum_{k \in I^d} (D_\nu^+)^s \sigma(kh) e^{ikjh} h^d = \beta(j_\nu h, h)^s a_j, \quad \beta = (e^{-ij_\nu h} - 1)/h \quad (\text{A.10})$$

so that  $D_\nu^+ \sigma(kh)$  is the transform of the last term, and thus by (A.2)

$$\sum_{j \in I^d} |\beta(j_\nu h, h)|^{2s} |a_j|^2 = (2\pi)^{-d} \sum_{k \in I^d} |(D_\nu^+)^s \sigma(kh)|^2 h^d. \quad (\text{A.11})$$

We note that  $|e^{-ij_\nu h} - 1| = 2|\sin(j_\nu h/2)| \geq c|j_\nu h|$  since  $|j_\nu h| \leq \pi$ . Thus  $\sum_j |j_\nu|^{2s} |a_j|^2$  is bounded by the right side of (A.11), and summing over  $\nu$  we get

$$\sum_{j \in I^d} |j|^{2s} |a_j|^2 \leq K \sum_{\nu=1}^d \sum_{k \in I^d} |(D_\nu^+)^s \sigma(kh)|^2 h^d. \quad (\text{A.12})$$

Finally, combining (A.6),(A.9),(A.12) gives the conclusion (A.4).  $\square$

*Proof of Lemma 2.1.* For simplicity we assume  $L = \pi$ . From (2.2), (2.3) the Fourier symbol of  $A$  in case  $d = 2$  is

$$\sigma(\xi) = (s_1^4 + s_2^4)/(s_1^2 + s_2^2), \quad s_j = \sin(\xi_j/2) \quad (\text{A.13})$$

where  $\xi = kh$ , and similarly for  $d = 3$ . We set  $\sigma(0) = 0$  so that  $A$  is extended to all of  $\mathcal{F}(\Omega_h)$ . For  $|\xi_j| \leq \pi$ ,  $|s_j| \geq c|\xi_j|$ , and  $\sigma$  and  $\partial\sigma/\partial\xi_j$  are bounded. It is easy to check that  $\partial^2\sigma/\partial\xi_j^2$  is bounded for  $\xi \neq 0$ . It follows that the second difference  $D_j^2\sigma$  is bounded at grid points not adjacent to 0. However, for grid points near 0,  $\sigma = O(h^2)$ , so that  $D_j^2\sigma$  is bounded in that case also. Thus the right side of (5.4) is bounded independent of  $h$  with  $s = 2$ , and Lemma A.1 ensures that the same is true for  $\|A\|$ .

## References

- [1] A. S. Almgren, J. B. Bell, and W. Y. Crutchfield, *Approximate projection methods: Part I. Inviscid analysis*, SIAM J. Sci. Comput. 22 (2000), 1139–59.
- [2] R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, Dover, New York, 1962.
- [3] J. T. Beale and A. T. Layton, *On the accuracy of finite difference methods for elliptic problems with interfaces*, Commun. Appl. Math. Comput. Sci. 1 (2006), pp. 91-119.
- [4] J. T. Beale, *Smoothing properties of implicit finite difference methods for a diffusion equation in maximum norm*, SIAM J. Numer. Anal. 47 (2009), pp. 2476-95.
- [5] P. Brenner, V. Thomée, and L. B. Wahlbin, *Besov spaces and applications to difference methods for initial value problems*, Lec. Notes in Math. 434, Springer, 1975.

- [6] D. Brown, R. Cortez and M. Minion, *Accurate projection methods for the incompressible Navier-Stokes equations*, J. Comput. Phys. 168 (2001), 464–99.
- [7] W. E and J.-G. Liu, *Projection method III. Spatial discretization on the staggered grid*, Math. Comp. 71 (2002), 27–47.
- [8] J. L. Guermond, P. Mineev and J. Shen, *An overview of projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg. 195 (2006), 6011–45.
- [9] R. D. Guy and A. L. Fogelson, *Stability of approximate projection methods on cell-centered grids*, J. Comput. Phys. 203 (2005), 517–38.
- [10] R. D. Guy and D. A. Hartenstine, *On the accuracy of direct forcing immersed boundary methods with projection methods*, J. Comput. Phys. 229 (2010), 2479–96.
- [11] W. Hackbusch, *On the regularity of difference schemes*, Ark. Mat. 19 (1981), 7195.
- [12] B. S. Jovanovic and E. Suli, *Analysis of finite difference schemes for linear partial differential equations with generalized solutions*, Springer, London, 2014.
- [13] D. Le, B. Khoo and J. Peraire, *An immersed interface method for viscous incompressible flows involving rigid and flexible boundaries*, J. Comput. Phys. 220 (2006), 109–38.
- [14] L. Lee and R. LeVeque, *An immersed interface method for incompressible Navier-Stokes equations*, SIAM J. Sci. Comput. 25 (2003), 832–56.
- [15] R. J. LeVeque and Z. Li, *Immersed interface methods for Stokes flow with elastic boundaries or surface tension*, SIAM J. Sci. Comput. 18 (1997), 709–35.
- [16] Z. Li and K. Ito, *The Immersed Interface Method*, SIAM, Philadelphia, 2006.
- [17] Z. Li and M.-C. Lai, *The immersed interface method for the Navier-Stokes equations with singular forces*, J. Comput. Phys. 171 (2001), 822–42.
- [18] J.-G. Liu, J. Liu, and R. L. Pego, *Stable and accurate pressure approximation for unsteady incompressible viscous flow*, J. Comput. Phys. 229 (2010), 3428–53.
- [19] X.-D. Liu and T. C. Sideris, *Convergence of the ghost fluid method for elliptic equations with interfaces*, Math Comp. 72 (2003), 1731–46.
- [20] A. Mayo, *The fast solution of Poisson’s and the biharmonic equations on irregular regions*, SIAM J. Numer. Anal. 21 (1984), 285–99.
- [21] K. W. Morton and D. Mayers, *Numerical Solution of Partial Differential Equations*, second ed., Cambridge Univ. Press, Cambridge, 2005.
- [22] C. Peskin, *The immersed boundary method*, Acta Numer. 11 (2002), 479–517.
- [23] C. Peskin and B. Printz, *Improved volume conservation in the computation of flows with immersed elastic boundaries*, J. Comput. Phys. 105 (1993), 33–46.



- [24] R. Stevenson, *Discrete Sobolev spaces and regularity of elliptic difference schemes*, RAIRO Modl. Math. Anal. Numr. 25 (1991), 607-40.
- [25] S. Xu and Z. J. Wang, *An immersed interface method for simulating the interaction of a fluid with moving boundaries*, J. Comput. Phys. 216 (2006), 454-93.
- [26] S. Xu and Z. J. Wang, *Systematic derivation of jump conditions for the immersed interface method in three-dimensional flow simulation*, SIAM J. Sci. Comput. 27 (2006), 1948-80.
- [27] A. Wiegmann and K. P. Bube, *The explicit-jump immersed interface method: finite difference methods for PDEs with piecewise smooth solutions*, SIAM J. Numer. Anal. 37 (2000), 827-62.